The iterated exponential

I asked

a) If $x^{x^{x^{\cdots}}} = 2$, what is x?

b) If $x^{x^{x^{\cdots}}} = 4$, what is x?

c) Compare the two sequences

$$a_0 = 1, \quad a_{n+1} = x^{a_n} \quad n = 0, 1, 2, \ldots$$

and

$$b_0 = 1, \quad b_{n+1} = b_n^{\,x} \quad n = 0, 1, 2, \ldots$$

That last had an error: it should have been

$$c_0 = x, \quad c_{n+1} = c_n^{\,x}, \quad n = 0, 1, 2, \ldots$$

(no matter: both sequences are boring)

Naive solution to (a) (a Grade 11 Math Contest problem from the 1970's)

If $x^{x^{x^{\cdots}}} = 2$, then $x^2 = 2$

so $x = \sqrt{2} = 1.4142\ldots$

This solution relies on recognizing the power of x in

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \quad \text{as just} \quad x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \quad \text{itself}$$

(only an infinite set can be identical to a proper subset of the set!)

An alternative solution uses logarithms:

$$\log\left(x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}\right) = \log 2$$

so by $\log(a^b) = b \log a$

where here $a = x$ and $b = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$

we have $b \log a = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \log x = \log 2$

or (using $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$) $\quad 2 \log x = \log 2$

$$\therefore \log x = \frac{1}{2} \log 2 = \log 2^{1/2}$$

and so $x = 2^{1/2} = \sqrt{2}$ as before.

We will see that this solution is correct.

Now we use the same trick on part (b).

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 4 \Rightarrow x^4 = 4$$

$$\Rightarrow x = 4^{1/4} = 2^{1/2} = \sqrt{2} \text{ again.}$$

La-de-dah, that was easy.

Ooops, though:

$$2 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}} = 4$$ which isn't so nice.

what went wrong?! We used the same method!

The problem is that there is <u>no</u> <u>such</u> <u>x</u> that $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 4$.

(Proving that is kind of hard, though).

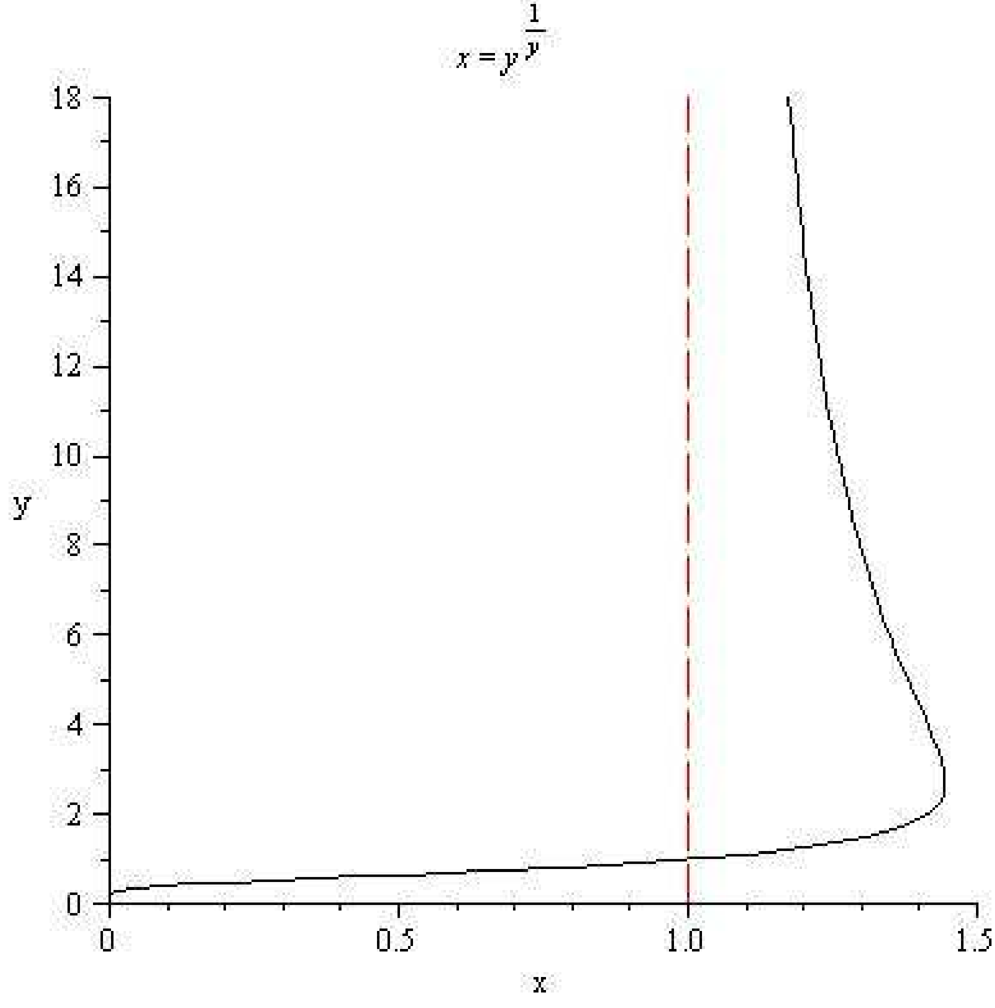
Before we look at the sequences,

consider the "general" problem

If $x^{x^{x^{\cdot^{\cdot}}}} = y$, what is x?

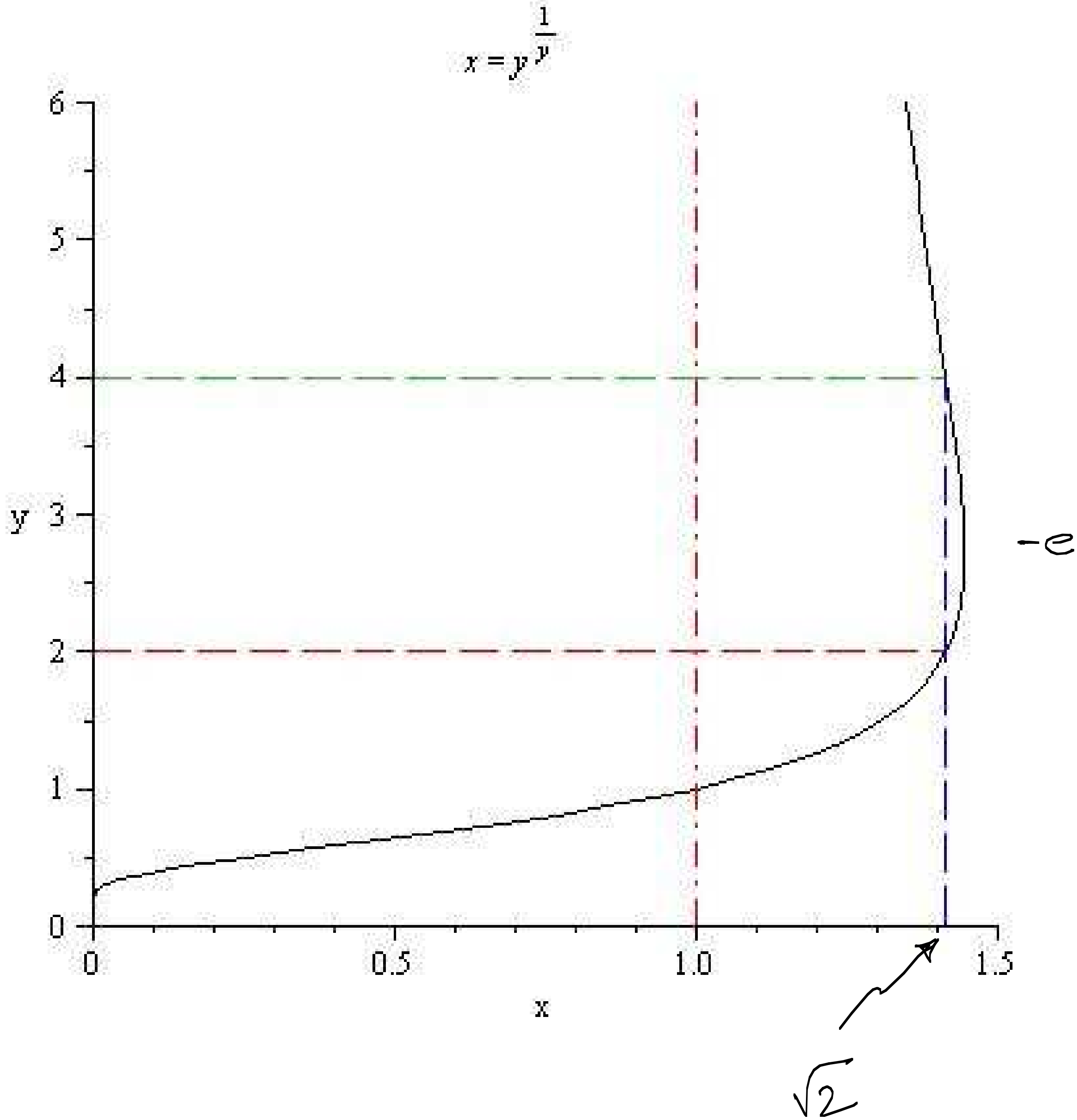
Proceeding as before (though we are now very suspicious)

$$x^y = y \quad \text{so} \quad x = y^{1/y}.$$



This graph plotted via Maple

In more detail,



This shows the coincidence of getting the same answer.

Now what about the sequences?

Take $x = \sqrt{2} = 1.41421356237\ldots$

on a calculator $a_{n+1} = 1.414\ldots^{a_n}$

n	a_n	
0	1.0000	
1	1.41421	
2	1.63253	$\doteq 1.4^{1.4}$
3	1.76084	$\doteq 1.4^{1.6}$
4	1.84091	$\doteq 1.4^{1.76}$
5	1.89271	etc
6	1.92700	
7	1.95003	
8	1.96566	
9	1.97634	
10	1.98367	
⋮		
20	1.99959	
⋮		
30	1.99999	

convincing evidence $\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}} = 2$,
not 4.

The other iterations are boring:

$b_0 = 1 \qquad b_1 = 1^x = 1$

$b_2 = 1^x = 1 \;,\; b_3 = \;\ldots\; b_n = 1.$

(goof)

$c_0 = x$ more interesting

$c_1 = x^x$ same as a_1

$c_2 = (x^x)^x = x^{x^2}$ $\underline{\underline{\text{not}}}$ a_2

$a_2 = x^{x^x}$ here $c_2 = x^{x^2}$

$c_3 = (x^{x^2})^x = x^{x^2 \cdot x} = x^{x^3}$

in general $c_n = x^{x^n}$

"flat tower" $(x)^{x \cdot x \cdot x \cdot x \cdots x}$

$$\lim_{n\to\infty} x^n = \begin{cases} \infty & \text{if } x>1 \\ 1 & \text{if } x=1 \\ 0 & \text{if } 0\le x<1 \end{cases}$$

So $C_n \to \infty$, 1, or 0 (?) (and is never 2 or 4).

(actually it's not so clear $x^{x^n} \to 0$ if $0<x<1$, but it does).

So the only reasonable meaning for $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ is

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} := \lim_{n\to\infty} a_n$$

if this limit exists.

Advanced A proof.

Theorem: (Euler) If $1 < x < e^{1/e}$ then this limit exists.

Proof: First we show $a_n \le e$ (where $e = 2.71828\ldots$ is the base of the natural logarithms) and then we show

$$a_0 < a_1 < a_2 < \cdots$$

so the sequence is increasing. Then by a property of real numbers we will have shown that the limit exists.

To show the first part, by induction, note $a_0 = 1 < e$.

Assume that $a_k \leq e$.

Then $$a_{k+1} = x^{a_k} \leq \left(e^{1/e}\right)^{a_k}$$

since x^{anything} is an increasing function of x, and so

$$a_{k+1} \leq \left(e^{1/e}\right)^{a_k} \leq \left(e^{1/e}\right)^{e}$$

since $(\text{anything} > 1)^{a_k}$ is an increasing function of a_k.

But $\left(e^{1/e}\right)^e = e^{\frac{1}{e}\cdot e} = e^1 = e$ and this shows $a_{k+1} \leq e$ if a_k is. Since $a_0 \leq e$, this implies $a_1 \leq e$, then likewise $a_2 \leq e$, and so on for all a_n.

Now we need to show

$$a_{k+1} > a_k.$$

Again we use induction. Note

$$a_1 = x^1 = x > a_0 = 1$$

because we assume $1 < x \le e^{1/e}$.

Assume then that

$$a_k > a_{k-1} \quad \text{for some } k>0.$$

Then $a_{k+1} = x^{a_k} > x^{a_{k-1}}$

because again this is an increasing function of a, since $x>1$.

But $x^{a_{k-1}} = a_k$, by definition.

$$\therefore \ a_{k+1} = x^{a_k} > a_k.$$

The induction completes the result.

The sequence thus converges if $1 < x \le e^{1/e}$. It converges to the lower part of the curve in the graph (we showed that $a_n \le e$, remember).

Hence $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ can never be equal to 4, if $1 < x$.

The case $x = 1$ is easy.

If $e^{-e} \leq x < 1$,

the sequence <u>also</u> converges.

(the proof is more complicated, though Euler knew it).

It is very surprising that the sequence does <u>not</u> converge if $0 < x < e^{-e} \doteq 0.066$

(a "stable two-cycle bifurcates from the unstable steady solution").

Finally:

$$y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

implies $\quad y^{1/y} = x$

if $\quad e^{-e} \le x \le e^{1/e}$

and this can be 'solved' in terms of the Lambert W function:

$$y = \frac{W(-\ln x)}{-\ln x}$$

See the Lambert W Poster.